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Quantum Mechanical Hamiltonian Models of Discrete Processes That Erase Their Own Histories: Application to Turing Machines

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Work done before on the construction of quantum mechanical Hamiltonian models of Turing machines and general discrete processes is extended here to include processes which erase their own histories. The models consist of three phases: the forward process phase in which a map T is iterated and a history of iterations is generated, a copy phase, which is activated if and only if T reaches a fix point, and an erase phase, which erases the iteration history, undoes the iterations of T, and recovers the initial state except for the copy system. A ballast system is used to stop the evolution at the desired state. The general model so constructed is applied to Turing machines. The main changes are that the system undergoing the evolution corresponding to T iterations becomes three systems corresponding to the internal machine, the computation tape, and computation head. Also the copy phase becomes more complex since it is desired that this correspond also to a copying Turing machine.

1. INTRODUCTION

In an earlier paper (Benioff, 1980) microscopic quantum mechanical Hamiltonian models of Turing machines were constructed. Successive steps in the model were initiated by a system scattering from a sequence of fixed scattering centers where the phase shifts from successive one-dimensional scatterings turned on successive step interactions. The Coleman model (Bell, 1975; Hepp, 1972), which makes the kinetic energy linear in the momentum and results in no wave packet spreading, was used for the moving system.

This work was recently generalized (Benioff, 1981) to cover all discrete processes represented as iterations of a map T on a countably infinite set A. The Coleman approximation was removed and a locality requirement was

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imposed. Some aspects of the relationship between the resulting models and the dilation theory of Foias and Sz. Nagy (Foias and Sz. Nagy, 1970) were discussed.

In both papers the Hamiltonian evolution generates a history of the process on a recording system. However as Landauer and others have emphasized (Keyes and Landauer, 1970; Landauer and Woo, 1971; Landauer, 1976), making the process reversible by creating history tapes only delays the inevitable. The history tapes have to be erased sometime.

Bennett (1973) has shown that one can associate to each Turing machine a reversible Turing machine which erases its own history. In particular, the machine calculates, and generates, a record of the calculation in the forward phase. If and when the machine terminates it copies the expression on the computation tape onto an extra copy tape. Then it proceeds to erase the record and recover the initial state before halting. Bennett discussed physical models of these machines which were thermodynamically reversible and thus dissipated arbitrarily little energy if they proceeded slowly enough.

Here Bennett's method will be applied as follows. For each discrete process $T: A \to A$ (T can be many-one), a quantum mechanical Hamiltonian model which is valid for $3n$ steps will be constructed. (n depends on adjustable system parameters and is otherwise arbitrary.) In the first phase the model carries out iterations of T on the A system and generates a history on the record system. If and only if the model arrives at a fix point of T in less than n iterations of T, the fix point is copied onto another system. Then the model enters the erase phase in which the record is erased as the T iterations on the A system are undone. If the system arrives at the initial state of the A system and a blank record system in less than $3n$ steps, then no further changes in the A system and the record system occur.

The model constructions are then applied to Turing machines. This requires expanding the A system to an internal machine system, a tape system, and a tape head. However, it is not necessary to record a complete machine "instantaneous description" into a record system cell at each recording step. Also, the copying phase becomes more complex.

The main reason for first considering models of abstract discrete processes is that they are relatively simple. Basic characteristics of the model are clearer and easier to see. Direct construction of quantum mechanical Hamiltonian models of Bennett's machines is quite messy and one tends to lose important points in the details. Proceeding in this way also shows that the model construction is not limited to Turing machines.

2. REVIEW OF TURING MACHINES

It is worthwhile at this point to give a brief review of Turing machines (Bennett 1973; Davis, 1958). For more details the reader is referred to the

literature. The reason Turing machines are considered here is that they give a standard representation of digital computers. Thus, the results obtained here apply to all digital computers.

A Turing machine consists of three parts, an internal system $\mathfrak{L}$, an infinite tape $\mathfrak{T}$, divided into cells at positions $\cdots -1,0,+1\cdots$, and a tapehead j, which scans the cells one at a time. Each cell may be blank or it may contain any one of a finite number of symbols in S, the symbol alphabet. The tap expression is given by the sequence of symbols, blanks included, on the tape. Since one is concerned with expressions which contain only a finite number of nonblank symbols, the infinite sequences of blanks at both ends of the expression are often suppressed to give expressions of finite length. $\mathfrak{L}$ can assume any one of a finite number of internal states. The set of such states can be different for different machines.

The Turing machine begins with an initial expression on the tape and proceeds by a sequence of discrete steps modifying the tape expression as it proceeds. Under a fixed correspondence between symbol sequences and natural numbers, each Turing machine Q defines a function f_Q on the natural numbers where f_Q is undefined at m if Q, started with m on the tape never halts. If Q halts, the value $f_Q(m)$ is given by the final number on the tape.

The basic operations of Turing machines are described as follows: The head j, at some position i, scans the ith cell of $\mathfrak{T}$. Depending on the state of $\mathfrak{L}$ and the symbol scanned, the symbol is changed and/or j moves one cell to the right or to the left. The state of $\mathfrak{L}$ can also change. These operations are represented by quintuples of the form $l(s, s'\sigma)m$, where l and s represent the initial state of $\mathfrak{L}$ and the symbol scanned by j. s' denotes the changed symbol in the cell ($s = s'$ is possible). σ represents the shift of j where $\sigma = -1, +1, 0$ denote, respectively, shift of j to the next cell on the left, or right, or no shift. m denotes the final state of $\mathfrak{L}$.

Each Turing machine is represented by a finite set of quintuples no two of which begin with the same first two symbols. From now on we let Q denote both a Turing machine and such a finite set of quintuples. The step succession is determined as follows: Let l and s denote the state of $\mathfrak{L}$ and the $\mathfrak{T}$ symbol scanned. Then the next operation is given by the quintuple in Q of the form $l(s,\text{--})\text{-}$. By the above definition, there is at most one such quintuple in Q. If no quintuple in Q begins with l and s [or there is one and it has the form $l(s, s0)l$], the process halts.

The overall Turing machine state, or instantaneous description, can be represented by a triple (l, ϕ, j) as an element of $L \times (S_b)_b^Z \times Z$. Here L denotes the set of all states of $\mathfrak{L}$, Z is the set of all integers or cell positions in $\mathfrak{T}$, and $(S_b)_b^Z$ is the set of all sequences ϕ: $Z \to S_b$, where $S_b = S \cup \{b\}$ and b denotes a blank and $\phi(j) \neq b$ for at most a finite number of j values. A triple (l, ϕ, j) denotes the state of $\mathfrak{L}$, the sequence of symbols (all blanks

included) on $\mathfrak{T}$, and the position of the scanned cell of $\mathfrak{T}$. Note that L is a sum over all machines of the states associated with each machine. Let $ID = L \times (S_b)^Z_b \times Z$. Then each Turing machine, Q, corresponds to a function T_Q: $ID \to ID$ defined as follows:

$$T_Q(l\phi j) = (l'\phi' j') \tag{1}$$

where $l'\phi' j'$ are given by that quintuple in Q which begins with l and $\phi(j)$. The quintuple can be represented by $l(\phi(j), s'\sigma)m$. This implies that $l' = m$, $j' = j + \sigma$, and $\phi'(j) = s'$ and $\phi'(k) = \phi(k)$ for all $k \neq j$. If no quintuple in Q begins with l and $\phi(j)$, then $l' = l$, $\phi' = \phi$, and $j' = j$. It is clear from the above that for each machine Q, there is exactly one transfer function T_Q and that T_Q is many-one in general. The steps of Q correspond to iterations of T_Q. Q halts at some (l, ϕ, j) if and only if (l, ϕ, j) is a fixpoint of T_Q. The set ID is countably infinite and represents the state space for all Turing machines.

It is convenient to restrict Turing machines to a standard representation of initial states and final states (Bennett, 1973). The process begins with $\mathfrak{L}$ in a fixed initial state l_i and with the tape head at position 0. The tape expression ϕ_i is such that all nonblank symbols (if any) start at $\phi_i(1)$ and extend in the direction of increasing cell position. If the computation halts, it halts with the tape head at position 0, $\mathfrak{L}$ in a fixed final state l_f, and ϕ_f on $\mathfrak{T}$, where $\phi_f(1)$ is the first nonblank symbol on $\mathfrak{T}$ (if any are present). Also ϕ_i and ϕ_f are such that no blanks occur between nonblank symbols. The initial and final steps of the machine are given by the quintuples $l_i(b, b+1)l_1$ and $l_{k-1}(s, s-1)l_f$ for each s in S.

The number of $\mathfrak{L}$ states attainable by any standard Turing machine, summed over all calculations, depends on the step number n and the number m of symbols in S_b. Two states l_i and l_1 suffice for the first step. For any given state, l_j there are at most m distinct quintuples in Q beginning with l_j and ending with different $\mathfrak{L}$ states. Iteration gives the result that after n steps have occurred, the number of distinct $\mathfrak{L}$ states attainable, summed over all calculations, is $\sum_{j=0}^{n-1} m^j + 1 = N_n$.

One further restricts standard machines Q so that, under a fixed numbering $l_1, l_2, \ldots$ of the states in L, the $\mathfrak{L}$ states attainable in the first n steps of any calculation lie in $L^n = \{l_1, l_2, \ldots, l_{N_n}\} U \{c, d\}$. The two states c and d are needed for the copy phase.

In this paper, consideration is limited to models which describe the first n steps of any standard Turing machine computation. The set of overall Turing machine states of relevance for the first n steps is given by the finite set $ID_n = L^n \times (S_b)^{\{-n, n\}} \times \{-n, n\}$. The interval $\{-n, n\}$ occurs because starting from 0, the tape head can shift at most n cells in either direction in n steps.

3. THE BASIC MODEL

3.1. Model Systems and State Descriptions. The construction of quantum mechanical Hamiltonian models of abstract discrete processes T on A will now be considered. The model includes the system $\mathfrak{A}$ on which model operations corresponding to T will occur, a record system $\mathfrak{R}$ which consists of a one-dimensional lattice of n record cell systems, and a recording head $\mathfrak{h}$ which moves along $\mathfrak{R}$ scanning the record cells. The model is completed by addition of a widely spaced lattice $\mathfrak{S}$ of $3n$ scattering centers and a system ω which moves along $\mathfrak{S}$ scattering from each of the centers in turn. The use of ω and $\mathfrak{S}$ will be described later. Figure 1 illustrates the setup. Here, script letters will denote either the systems or the quantum systems in the model. It will be clear from context which is meant.

The complete state description of $\mathfrak{A}$ is given in terms of a complete set $\{\psi_a | a \in \mathfrak{A}\}$ of quantum states which are pairwise orthogonal and normalized to 1 (an orthonormal basis). $\psi_a^{\mathfrak{A}}$ is the quantum system model equivalent of the state a of $\mathfrak{A}$. For each record cell system $\mathfrak{r}_j$, an appropriate orthonormal basis is $\{\psi_\gamma^{\mathfrak{r}_j} | \gamma \in A_b\}$, where $A_b = AU\{b\}$. The blank symbol b is needed since a blank record cell must be distinguished from all recorded A elements. The record system $\mathfrak{R}$ is completely described by an orthonormal basis set $\{\psi_\gamma^{\mathfrak{R}} | \gamma \in A_b^n\}$. The model quantum state $\psi_\gamma^{\mathfrak{R}} = \otimes_{j=1}^n \psi_{\gamma(j)}^{\mathfrak{r}_j}$ corresponds to the expression $\gamma(1), \gamma(2), \ldots, \gamma(n)$ recorded into the record system where the ith cell is blank if $\gamma(i) = b$. the recording head system $\mathfrak{h}$ is described in terms of an orthonormal basis set $\{\psi_k^{\mathfrak{h}} | k = 1, \ldots, n\}$. The state $\psi_k^{\mathfrak{h}}$ corresponds to the head being at cell position k.

The state of the $\mathfrak{A} + \mathfrak{R} + \mathfrak{h}$ system which corresponds to $\mathfrak{A}$ in state a, expression γ recorded in the cells of $\mathfrak{R}$ and $\mathfrak{h}$ at position k, $\Psi_{a\gamma k}$ is given as the tensor product of the component states or

$$\Psi_{a\gamma k} = \psi_a^{\mathfrak{A}} \otimes \psi_\gamma^{\mathfrak{R}} \otimes \psi_k^{\mathfrak{h}} \tag{2}$$

The process is started with $\mathfrak{A} + \mathfrak{R} + \mathfrak{h}$ in an initial state of the form $\Psi_{a\mathbf{b}1}$ given by equation (2). Here a is an arbitrary element of $\mathfrak{A}$ and $\mathbf{b}$ denotes the constant blank sequence on $\mathfrak{R}$. The state $\Psi_{a\mathbf{b}1}$ corresponds to $\mathfrak{A}$ in state a, all cells in $\mathfrak{R}$ blank, and $\mathfrak{h}$ at cell position 1.

The goal of the process is to construct a Hamiltonian for the model system so that under the Schrödinger equation, the overall system evolves so that the $\mathfrak{A} + \mathfrak{R} + \mathfrak{h}$ system proceeds through an iteration of three types of steps repeated over and over. The first type of step records the state a of $\mathfrak{A}$ into the (blank) cell of $\mathfrak{R}$ scanned by $\mathfrak{h}$. The second type carries out on $\mathfrak{A}$ the state change $a \to T(a)$ corresponding to the value of a in the record cell scanned by $\mathfrak{h}$. The third type of step shifts $\mathfrak{h}$ to the next record cell on the right.

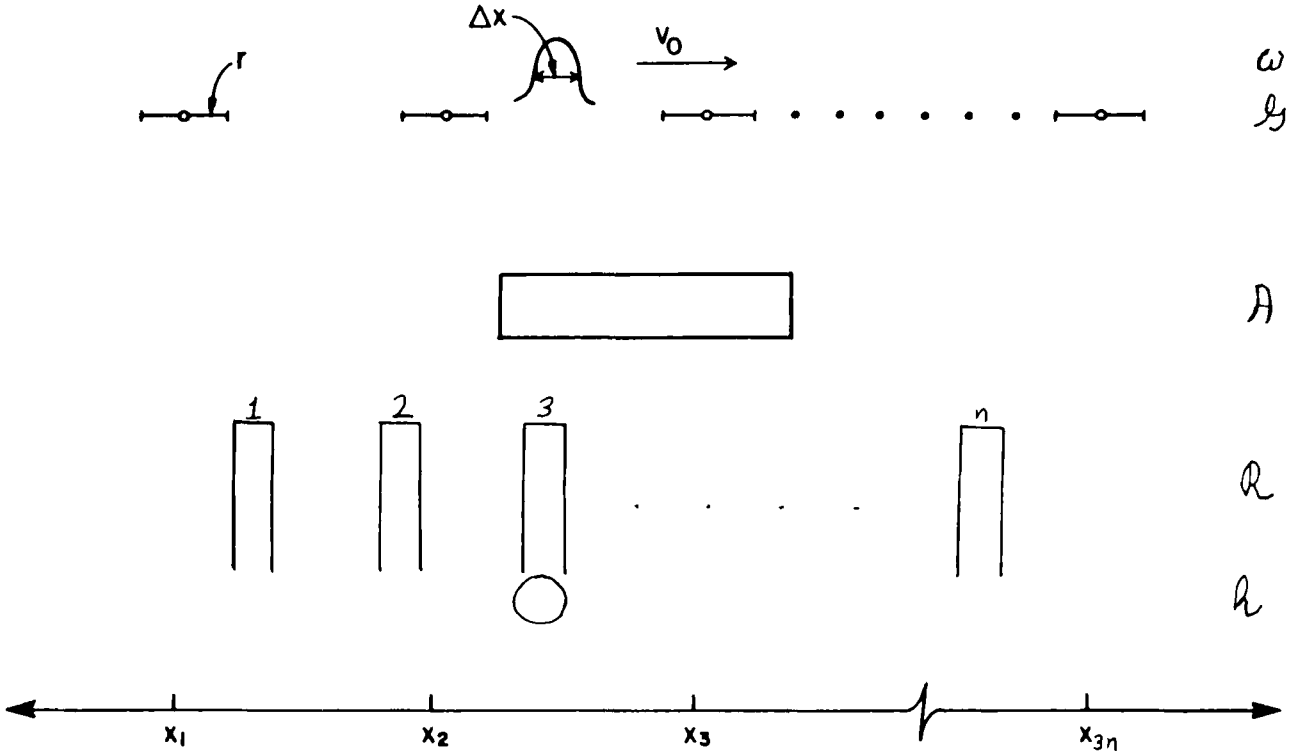

Fig. 1. A schematic representation of the basic model. $\mathfrak{G}$ denotes the row of fixed scattering centers, ω denotes the particle with group velocity v_0 and wave packet width Δx which scatters from the centers in $\mathfrak{G}$. The range r of the interaction potential is denoted by a line segment at each of the scattering centers positioned at $x_1, \ldots, x_{3n}$. $\mathfrak{A}$, $\mathfrak{R}$, and $\mathfrak{h}$ denote the object systems with each record cell system in $\mathfrak{R}$ shown as an open-ended rectangle and the recording head $\mathfrak{h}$ shown by a circle.

The desired evolution of the $\mathfrak{A} + \mathfrak{R} + \mathfrak{h}$ system state is as follows: let V_1, V_2, and V_3 denote operators for the first, second, and third types of steps, respectively. Then for times t such that $3m + j$ model steps have been completed, where $j = 0$, 1, or 2 and $3m + j \leqslant 3n$, the $\mathfrak{A} + \mathfrak{R} + \mathfrak{h}$ system state, $\Psi_{ab!}(3m + j)$, should be given by

$$\Psi_{ab!}(3m + j) = W_j(V_3V_2V_1)^m\Psi_{ab!} \tag{3}$$

where $W_j = 1$ if $j = 0$, V_1 if $j = 1$, and V_2V_1 if $j = 2$. Since the model is supposed to be valid for n T iteration steps only, the above holds for the first $3n$ model steps only.

It is necessary to construct three interaction operators H_1, H_2, and H_3 which satisfy

$$V_j = \exp[iKH_j] \tag{4}$$

for $j = 1, 2, 3$. The interaction part of the overall system Hamiltonian will be constructed from H_1, H_2, and H_3. Details of the mathematical definitions of V_j and H_j are given in the next section. It may be omitted at first reading.

3.2. Model Operators. The record step operator V_1 is given by

$$V_1 = \sum_{a \in A} \sum_{k=1}^{n} P_a^{\mathfrak{A}} \otimes U_{ak}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \tag{5}$$

Here $P_a^{\mathfrak{A}}$ and $P_k^{\mathfrak{h}}$ are the projection operators for finding $\mathfrak{A}$ in state $\psi_a^{\mathfrak{A}}$ and $\mathfrak{h}$ in state $\psi_k^{\mathfrak{h}}$, respectively. $U_{ak}^{\mathfrak{R}}$ is the unitary operator which exchanges the kth component states, $\psi_a^{\mathfrak{r}\mathfrak{f}}$ and $\psi_b^{\mathfrak{r}\mathfrak{f}}$, only in $\psi_\gamma^{\mathfrak{R}}$. That is, $U_{ak}^{\mathfrak{R}}\psi_\gamma^{\mathfrak{R}} = \psi_{\gamma'}^{\mathfrak{R}}$ where $\gamma = \gamma'$ unless $\gamma(k) = a$ or b. In this case $\gamma'(k) = a$ if $\gamma(k) = b$ and $\gamma'(k) = b$ if $\gamma(k) = a$ and $\gamma(j) = \gamma'(j)$ for all $j \neq k$. Note that since $U_{ak}^{\mathfrak{R}}$ is an exchange operator $(U_{ak}^{\mathfrak{R}})^2 = 1$.

It is clear from the above definition that V_1 acting on any state of the form $\Psi_{a\gamma k}$, Eq. (2), where $\gamma(k) = b$, records a into the kth cell of γ. In this case $V_1\Psi_{a\gamma k} = \Psi_{a\gamma' k}$, where $\gamma'(k) = a$ and $\gamma(j) = \gamma'(j)$ at all $j \neq k$. Note that V_1 acting on a state of the form $\Psi_{a\gamma k}$ where $\gamma(k) = a$ is an erasing operator. This will be of use later on. Also, V_1 does not change $\Psi_{a\gamma k}$ if $\gamma(k) \neq b, a$.

The T iterate operator V_2 is given by

$$V_2 = \sum_{\gamma \in A_R} \sum_{k=1}^{n} U_\gamma^{\mathfrak{A}} \otimes P_{\gamma k}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \tag{6}$$

Here $P_{yk}^{\mathfrak{R}}$ is the projection operator for finding $\mathfrak{R}$ in the state ψ_γ with $\gamma(k)=y$ and $\gamma(j)$ arbitrary for $j\neq k$ (i.e., the projection operator for finding y in cell k of $\mathfrak{R}$). $P_k^{\mathfrak{h}}$ is as defined above. $U_y^{\mathfrak{a}}$ is the operator which exchanges the states $\psi_y^{\mathfrak{a}}$ and $\psi_{T(y)}^{\mathfrak{a}}$ and leaves all other states in the basis set $\{\psi_a^{\mathfrak{a}} \mid a\in A\}$ alone. $U_y^{\mathfrak{a}}$ is the identity if either $y=b$ or $T(y)=y$. For each y, $U_y^{\mathfrak{a}}$ is unitary and $(U_y^{\mathfrak{a}})^2=1$.

V_2, acting on any state of the form $\Psi_{a\gamma k}$ where $\gamma(k)=a$, iterates T on $\mathfrak{a}$. That is, in this case, it converts $\Psi_{a\gamma k}$ to the state $\Psi_{T(a)\gamma k}$. However, V_2 acting on the state $\Psi_{T(a)\gamma k}$ with $\gamma(k)=a$, reverses the T iteration converting the state to $\Psi_{a\gamma k}$. V_2 does not change $\Psi_{a\gamma k}$ if $a\neq\gamma(k)$ and $T(\gamma(k))\neq a$.

The head shift operator V_3 is given by

$$V_3=\sum_{k=1}^{n} 1^{\mathfrak{a}}\otimes P_k^{\mathfrak{R}}\otimes U_{k+1}^{\mathfrak{h}}+1^{\mathfrak{a}}\otimes P_{\mathbf{b}}^{\mathfrak{R}}\otimes 1^{\mathfrak{h}} \tag{7}$$

Here $P_k^{\mathfrak{R}}$ is the projection operator for all states ψ_γ such that $\gamma(k)\neq b$ and $\gamma(j)=b$ for all $j>k$. That is, $P_k^{\mathfrak{R}}$ projects out all record states in which cell k is the last (in the direction of increasing k) nonblank cell. $P_{\mathbf{b}}^{\mathfrak{R}}$ is the projection operator for the constant blank sequence [$\gamma(j)=b$ for all j] and the 1's denote identity operators. $U_{k+1}^{\mathfrak{h}}$ is the operator which exchanges the states $\psi_k^{\mathfrak{h}}$ and $\psi_{k+1}^{\mathfrak{h}}$ modulo–and leaves all other states in the basis set alone. Modulo–means that if $k=n$ then $\psi_{k+1}^{\mathfrak{h}}=\psi_1^{\mathfrak{h}}$. For each k, $U_{k+1}^{\mathfrak{h}}$ is unitary and $(U_{k+1}^{\mathfrak{h}})^2=1$.

V_3, acting on any state of the form $\Psi_{a\gamma k}$ where $\gamma(k)\neq b$, changes it to $\Psi_{a\gamma k+1}$ modulo —. That is, it shifts $\mathfrak{h}$ one record cell to the right (Fig. 1). However V_3, acting on any state of the form $\Psi_{a\gamma k}$ where $\gamma(k)=b$ and $\gamma(k-1)\neq b$ ($k-1=n$ if $k=1$) converts the state to $\Psi_{a\gamma k-1}$. That is, it shifts $\mathfrak{h}$ one record cell to the left. V_3 does not change $\Psi_{a\gamma k}$ if γ and k do not have the relationship given above.

As defined the three-step operators V_1, V_2, V_3 are unitary. They are also exchange operators and self adjoint as $(V_j)^2=1$ for $j=1,2,3$. The terms in the sums over a, k, and y in equations (5)–(7) are all pairwise orthogonal (also $P_k^{\mathfrak{R}}P_{\mathbf{b}}^{\mathfrak{R}}=0$). In fact V_1, V_2, and V_3 were carefully constructed to make this condition hold. This is the main reason the recording head system is present and for the presence of the $P_k^{\mathfrak{R}}$ operator in equation (7), also without it V_1 would not be unitary.

As a consequence of the pairwise orthogonality, one can write

$$H_1=\sum_{a\in A}\sum_{k=1}^{n} P_a^{\mathfrak{a}}\otimes H_{ak}^{\mathfrak{R}}\otimes P_k^{\mathfrak{h}} \tag{8}$$

$$H_2=\sum_{a\in A}\sum_{k=1}^{n} H_a^{\mathfrak{a}}\otimes P_{ak}^{\mathfrak{R}}\otimes P_k^{\mathfrak{h}} \tag{9}$$

and

$$H_3 = \sum_{k=1}^{n} 1^{\mathfrak{A}} \otimes P_k^{\mathfrak{R}} \otimes H_{k+1}^{\mathfrak{h}} \tag{10}$$

where $H_{ak}^{\mathfrak{R}}$, $H_a^{\mathfrak{A}}$, and $H_{k+1}^{\mathfrak{h}}$ satisfy $U_a^{\mathfrak{A}} = \exp(iKH_a^a)$, $U_{ak}^{\mathfrak{R}} = \exp(iKH_{ak}^{\mathfrak{R}})$, and $U_{k+1}^{\mathfrak{h}} = \exp(iKH_{k+1}^{\prime})$. Equations (8)–(10) show that H_1, H_2, and H_3 can be explicitly given in terms of sums over tensor products of simpler operators on the component systems. Without the pairwise orthogonality it is not clear how to express H_1, H_2, and H_3 in terms of simpler component operators.

There are many possible choices for the elementary interaction operators $H_a^{\mathfrak{A}}$, $H_{ak}^{\mathfrak{R}}$, and $H_{k+1}^{\mathfrak{h}}$. Here these operators will all be taken to have the generic form

$$H = (\pi/K)(a^0 P^0 + a^1 P^1) \tag{11}$$

where a^0 and a^1 are constants. For $j = 0, 1$

$$P^j = \Psi^j)(\Psi^j \tag{12}$$

where

$$\Psi^j = \frac{1}{\sqrt{2}}\left[\Psi_1 + (-1)^j \Psi_2\right] \tag{13}$$

For $H_a^{\mathfrak{A}}$, $\Psi_1 = \psi_a^{\mathfrak{A}}$ and $\Psi_2 = \psi_{T(a)}^{\mathfrak{A}}$ if $T(a) \neq a$ [$H_a = 0$ if $a = T(a)$]. For $H_{ak}^{\mathfrak{R}}$, $\Psi_1 = \psi_b^{\mathfrak{r}_k}$ and $\Psi_2 = \psi_a^{\mathfrak{r}_k}$. Thus $H_{ak}^{\mathfrak{R}}$ acts on the system $\mathfrak{r}_k$ only of $\mathfrak{R}$. For $H_{k+1}^{\mathfrak{h}}$, $\Psi_1 = \psi_k^{\mathfrak{h}}$ and $\Psi_2 = \psi_{k+1}^{\mathfrak{h}}$ modulo n. These choices of $H_a^{\mathfrak{A}}$, $H_{ak}^{\mathfrak{R}}$, and $H_{k+1}^{\mathfrak{h}}$ are discussed in more detail elsewhere (Benioff, 1981).

3.3. The Scattering Systems ω and $\mathcal{G}$. The reason for the presence of the scattering systems ω and $\mathcal{G}$ is best seen by considering a simple example. Suppose one starts with a single spin-1/2 system with the spin aligned along the positive z axis (+), and wants to change the spin so it is aligned along the negative z axis (−). A simple way to do this is to impose a transverse magnetic field, of strength B, say, along the x axis. The appropriate Hamiltonian is $B\sigma_1$ where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ is the spin flip operator.

One way to proceed is to turn on the Hamiltonian $B\sigma_1$ at time $t = 0$ and turn it off at time $t = \pi/2B$ as $\exp(iB\sigma_1 t) = i\sigma_1$ if $t = \pi/2B$ and $\sigma_1 \psi_+ = \psi_-$. If one does not turn off the Hamiltonian it will flip the spin back from − to + and continue to cycle the spin state between ψ_+ and ψ_- with period $\pi/2B$.

The difficulty with the above is that the Hamiltonian is time dependent in that it must be turned on and off by an external agent. One can, however, construct a time-independent Hamiltonian by including into the model a scattering system ω and a fixed scattering center $\mathfrak{g}$ as such an agent. The model works in essence as follows. The Hamiltonian becomes $V(x)\sigma_1$ where $V(x)$ is the interaction potential between ω and $\mathfrak{g}$. The state of the spin-1/2 system at time t is given by $\exp[ip(t)\sigma_1]\psi_+$ where $p(t)$ is the total $\omega - \mathfrak{g}$ scattering phase shift at time t. [The scattering parameters can be adjusted so that the dependence of $p(t)$ on them is small.] One adjusts the strength and range of V so that $p(\infty) = \pi/2$ and $p(t) \sim p(\infty)$ for all t greater than some convenient time t_0. Also $p(t) = 0$ for times for which the ω wave packet has not yet reached $\mathfrak{g}$.

For the above Hamiltonian, the spin-1/2 system starts in state ψ_+. As ω starts interacting with $\mathfrak{g}$, $p(t)$ increases and the spin-1/2 system state becomes a linear superposition of ψ_+ and ψ_-. At time t_0 and for all times thereafter, the system state becomes ψ_- and remains ψ_-.

This method is easily extended to a multiple-step process by replacement of $\mathfrak{g}$ by a lattice $\mathfrak{S}$ of fixed widely separated scattering centers. The ω-particle scattering from each center in $\mathfrak{S}$ turns on an interaction Hamiltonian with a strength sufficient to complete one process step. The successive scatterings turn on successive step interaction operators and move the object system state through successive step changes [equation (3)] as ω progresses down the $\mathfrak{S}$ lattice.

In what follows the above is applied to the process of interest where $\mathfrak{P} + \mathfrak{R} + \mathfrak{h}$ is the object system. The scattering is limited to one-dimensional scattering to keep the discussion simple. The next section gives some mathematical details of the above and can be omitted at first reading.

3.4. The Overall System Evolution. The $\omega + \mathfrak{S} + \mathfrak{P} + \mathfrak{R} + \mathfrak{h}$ system Hamiltonian is given by $H = H_0 + H'$ where

$$H_0 = -\frac{\hbar^2}{2m}\nabla_x^2 \tag{14}$$

and

$$H' = \sum_{j=1}^{3n} V(x - x_j)H_{j \bmod 3} \tag{15}$$

with H_1, H_2, and H_3 given by equations (8)–(10). (We set $3l \bmod 3 = 3$ for all l.) Here H_0 is the free system Hamiltonian which describes the kinetic energy of ω, $V(x - x_j)$ is the one-dimensional interaction between ω and the jth center (at position x_j) of $\mathfrak{S}$, and $j \bmod 3$ denotes j modulo 3. There are

no terms in H_0 for the $\mathfrak{A}$, $\mathfrak{R}$, and $\mathfrak{h}$ systems as it is assumed here for simplicity that all states of $\mathfrak{A}$, $\mathfrak{R}$, and $\mathfrak{h}$ have zero energy in the absence of H'.

The overall system state at the time t and ω-position x is given by

$$\Psi(x,t)=\int \exp^{(-iE_k t/\hbar)}\phi(k-k_0)\Psi_{+k}(x)\,dk \tag{16}$$

where $E_k=\hbar^2k^2/2m$ is the total system energy and $\phi(k-k_0)$ denotes the ω system momentum space wave packet centered at k_0 and with spread Δk. $\Psi_{+k}(x)$ denotes the scattering solution for the Hamiltonian of equations (14) and (15) with incoming plane waves $\exp ikx$ and for the appropriate initial state of $\mathfrak{A}+\mathfrak{R}+\mathfrak{h}$. The spacing, d, of adjacent $\mathfrak{S}$ systems is such that ω interacts with at most one system at any one time. That is, $d>\Delta x+2r$ where Δx is the ω packet spread and r is the range of V about any center of $\mathfrak{S}$.

Under the eikonal approximation (Newton, 1966; Schiff, 1968) which requires that $V(x)$ change little over a distance $1/k$, $\Psi_{+k}(x)$ is given by (Benioff, 1981)

$$\Psi_{+k}(x)=\exp\left[ikx-\frac{im}{\hbar^2k}\int_{-\infty}^{x}V(x'-x_{n(x)})\,dx'H_{n(x)\bmod 3}\right]$$
$$\times\exp\left(\frac{iD}{k}H_{(n(x)-1)\bmod 3}\right)\cdots\exp\left(\frac{iD}{k}H_1\right)\Psi_{ab1} \tag{17}$$

where Ψ_{ab1} is the initial $\mathfrak{A}+\mathfrak{R}+\mathfrak{h}$ state, H_1, H_2, and H_3 are given by equations (8)–(10), and $n(x)$ is the label of the $\mathfrak{S}$ center which is either interacting with or has just completed interacting with ω system at x. D is given by

$$D=\frac{m}{\hbar^2}\int_{-r}^{r}V(x)\,dx=\frac{m}{\hbar^2}\int_{-\infty}^{\infty}V(x)\,dx \tag{18}$$

as $V(x)=0$ if $|x|>r$. m is the mass of ω. The strength of V is adjusted so that $D/k_0=K$. Note that $\exp[(iD/k_0)H_{j\bmod 3}]=V_{j\bmod 3}$. Details of the derivation are given elsewhere (Benioff, 1981).

The final step in the derivation (made implicitly in earlier work) is the replacement of equation (17) by

$$\Psi_{+k}(x)\simeq\exp\left[ikx-\frac{im}{\hbar k_0}\int_{-\infty}^{x}V(x'-x_{n(x)})\,dx'H_{n(x)\bmod 3}\right]$$
$$\times\exp(iKH_{(n(x)-1)\bmod 3})\cdots\exp(iKH_1)\Psi_{ab1} \tag{19}$$

This replacement is valid provided the momentum spread Δk is sufficiently narrow. In particular, writing $\exp(iDH_l/k) = \exp(iKH_l)\cdot\exp(iD\Delta kH_l/kk_0) \simeq \exp(iKH_l)\cdot(1+iK\Delta kH_l/k_0+\cdots)$ one sees that the approximation is valid for $3n$ steps provided that

$$3n < \frac{k_0}{K\Delta k \,\|H'\|} \tag{20}$$

where $\|H'\|$ = maximum of $\|H_1\|$, $\|H_2\|$, $\|H_3\|$. Use of equation (19) in (16) gives one the desired final model state which corresponds to equation (3).

3.5. Conditions on System Parameters. Equation (20) gives one condition which must be satisfied by the system parameters in order that the deviation of the phase shift D/k from K be small. Another condition is that the w system wave packet does not spread appreciably during the evolution of $3n$ model steps. This condition gives (Benioff, 1981)

$$3n < \frac{\sqrt{2}\,k_0\Delta x}{d\Delta k} \tag{21}$$

which expresses the fact that the wave packet must not spread enough to give interference in the scatterings from the different centers of $\mathcal{G}$.

Another requirement is that the eikonal approximation hold for the $3n$ potential scatterings, or $3nV_0/E_k < 1$. (A similar result is obtained for the related requirement that the transmission coefficient for the ω system wave packet through $3n$ square well potentials is close to unity.) This inequality gives

$$3n < rk_0/K \tag{22}$$

The above conditions show that the total number, $3n$, of steps can be made as large as desired by making the potential range r sufficiently large and the momentum dispersion Δk sufficiently small. In particular, one can set $r \sim \Delta x \sim 1/\Delta k$ and let r increase without limit. However, one must pay a price for such an increase in that as r and/or Δx increases so does the time required to carry out one step of the calculation. In particular, the time required to carry out one step of the calculation is equal to $md/\hbar k_0$ where $d > \Delta x + 2r$. The total evolution time during which the system evolves in the desired fashion is also bounded by $\sqrt{2}\,m\Delta x/(\hbar\Delta k)$, equation (21), or $md/(\hbar\Delta k\,\|H\|)$, equation (20), or $2mrd/(\hbar K)$, equation (22).

Finally, it is to be noted that the above limitations and requirements refer to the ω system scatterings from the centers in $\mathcal{G}$. No account has been

taken of limitations, if any, arising from the interactions between $\mathfrak{A}$, $\mathfrak{R}$, and $\mathfrak{h}$ given by H'.

It is of interest to consider a specific case to see what the effect of the above limitations is on system parameters. Let ω be a 0.25 MeV proton ($k_0 = 10^{12}$ cm^{-1}) with a momentum spread $\Delta k = 10^5$ cm^{-1}. Let $r = 10^{-5}$ cm, $\|H\| = K = 1$, and $d = 10^{-4}$ cm. Then the three conditions are satisfied with $3n = 10^6$. This gives a total $\mathfrak{S}$ lattice length of 1 m. Each model step is carried out in 10^{-13} sec.

4. MODELS WHICH ERASE THEIR OWN HISTORIES

The previous development can be used to construct models which erase their own histories. Bennett's construction (Bennett, 1973) for Turing machines will be followed in that before the erasure phase starts, the state of $\mathfrak{A}$ will be copied onto another system. Also the erasure phase will consist of steps similar to those of the forward phase but carried out in inverse order.

The model will proceed as follows: If T, started on a, either has no fix point or takes more than n iterations to arrive at the fix point, then the model remains in the forward phase for all $3n$ steps. If a T fix point, $\bar{a}$, is arrived at in less than n iterations of T, then the model process copies $\bar{a}$ onto a copy system and starts the erasure phase. This phase should erase the history from $\mathfrak{R}$ and thus return it to its initial blank state and also return $\mathfrak{A}$ to its initial state and $\mathfrak{h}$ to its initial position.

It turns out that such an expanded model can be constructed from the one already discussed. The main point to note is that the operators $V_1 - V_3$ are exchange operators. In particular, as discussed in Section 3.2, V_1 functions either as a recording or erasing operator, V_2 as a T or a T inverse operator, and V_3 as a right-shift or left-shift operator. Which mode these operators are in depends on the states on which they act.

The model is expanded by addition of a copy system $\mathfrak{r}$ to $\mathfrak{A} + \mathfrak{R} + \mathfrak{h}$ where the copy system is identical to a record cell system in $\mathfrak{R}$. The quantum states $\psi_y^{\mathfrak{r}}$ of $\mathfrak{r}$ for any y in A_b correspond to y copied on $\mathfrak{r}$. Let $P_b^{\mathfrak{r}}$ be a projection operator which projects out the blank state $\psi_b^{\mathfrak{r}}$ and define $P_{\neq b}^{\mathfrak{r}}$ by $1 - P_b^{\mathfrak{r}}$.

The model will be started with $\mathfrak{r}$ initially in a blank state. Thus whether or not $\mathfrak{r}$ is blank is used to determine which phase the model is in. To this end define interaction operators H_1' and H_3' by

$$H_1' = H_1 \otimes P_b^{\mathfrak{r}} + H_3 \otimes P_{\neq b}^{\mathfrak{r}} \tag{23}$$

and

$$H_3' = H_3 \otimes P_b^{\mathfrak{r}} + H_1 \otimes P_{\neq b}^{\mathfrak{r}} \tag{24}$$

It follows from equation (4) that if V_1' and V_3' are defined by $\exp(iKH_1')$ and $\exp(iKH_3')$, respectively, then

$$V_1' = V_1 \otimes P_b^{\mathfrak{r}} + V_3 \otimes P_{\neq b}^{\mathfrak{r}} \tag{25}$$

and

$$V_3' = V_3 \otimes P_b^{\mathfrak{r}} + V_1 \otimes P_{\neq b}^{\mathfrak{r}} . \tag{26}$$

For each a in $\mathfrak{q}$ let $U_a^{\mathfrak{r}}$ be the "copy operator." $U_a^{\mathfrak{r}}$ exchanges $\psi_a^{\mathfrak{r}}$ and $\psi_b^{\mathfrak{r}}$ and leaves all other states in the $\mathfrak{r}$ basis alone. Also, let $H_a^{\mathfrak{r}}$ satisfy $U_a^{\mathfrak{r}} = \exp(iKH_a^{\mathfrak{r}})$.

In order to construct the copying step operator, one needs a projection operator onto state(s) of $\mathfrak{q} + \mathfrak{R} + \mathfrak{h}$ which signifies when the copying is to be done. This is conveniently given by the projection operator $\Sigma_k \Sigma_a P_{ak-1}^{\mathfrak{R}} P_{ak}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}}$. The appropriateness of this operator stems from the fact that the same $\mathfrak{q}$ state label is recorded into adjacent record cells if and only if the label is a fix point of T. For any fix point $\bar{a}$ arrived at in the model process, the first time the record system will contain $\bar{a}$ in two adjacent cells is at the conclusion of a record step. Thus it is appropriate to modify H_2 to include the copying interaction by setting

$$H_2' = H_2^{\alpha} + H_2^{\beta} \tag{27}$$

where [equation (9)]

$$H_2^{\alpha} = \sum_{\substack{y \in A_b \\ y \neq a}} \sum_{a \in A} \sum_{k=1}^{n} H_a^{\mathfrak{q}} \otimes P_{yk-1}^{\mathfrak{R}} P_{ak}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \otimes 1^{\mathfrak{r}} \tag{28}$$

and

$$H_2^{\beta} = \sum_{a \in A} \sum_{k=1}^{n} 1^{\mathfrak{q}} \otimes P_{ak-1}^{\mathfrak{R}} P_{ak}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \otimes H_a^{\mathfrak{r}} \tag{29}$$

H_2^{α} is the part that acts in the forward or erase phase to carry out the model iterations or inverse iterations of T provided that the record cell scanned has a different symbol than the one adjacent on the left. The sum of y over A_b is necessary because after the first recording step, when the $k = 1$ term is active, $k - 1 = n$ and cell n of the record is blank.

H_2^{β} becomes active and copies the $\mathfrak{q}$ state into $\mathfrak{r}$ provided the record cell scanned by $\mathfrak{h}$ and the one immediately to the left both have the same a value recorded. Note that H_2^{α} and H_2^{β} are orthogonal.

The $\omega + \mathfrak{S} + \mathfrak{A} + \mathfrak{R} + \mathfrak{h} + \mathfrak{r}$ system Hamiltonian is given as $H_0 + H''$ where H_0 is given as before by equation (14) and

$$H'' = \sum_{j=1}^{3n} V(x - x_j) H'_{j \bmod 3} \tag{30}$$

The overall system state is given as before by equations (16)–(19) except that H' replaces H everywhere in equations (17) and (19). The initial states of interest for $\mathfrak{A} + \mathfrak{R} + \mathfrak{h} + \mathfrak{r}$ have the form $\Psi_{ab1b} = \Psi_{ab1} \otimes \psi_b^{\mathfrak{r}}$ with Ψ_{ab1} given by equation (2).

It is left to the reader to follow through in detail the evolution given by H'' to see that it has the desired behavior. Suffice it to note here that initially, only the H_1 part of H'_1, the H_2^{α} part of H'_2, and the shift part H_3 of H'_3 are active and function just as desired in Section 3.2.

H_2^{β} becomes active and copies the state of $\mathfrak{A}$ into $\mathfrak{r}$ if and only if the system arrives, at the end of a recording step, at a state $\Psi_{\bar{a}\gamma k b}$ where $\bar{a}$ is a fix point of T and $\gamma(k) = \gamma(k-1) = \bar{a}$. H_2^{β} converts the above state to $\Psi_{\bar{a}\gamma k \bar{a}}$. The erase phase now starts with the H_1 term in H'_3 first becoming active. Then the H_3 term in H'_1 becomes active and is followed by H_2^{α} being active. This sequence is repeated over and over. The overall system state is such that H'_3 functions as a record erase operator, H'_2 as a T-inverse operator and H'_1 as a left-shift operator. H_2^{β} is no longer active.

The system proceeds in this manner through the erase phase until $3n$ steps have occurred or until it arrives at the state $\Psi_{ab1\bar{a}}$. $\mathfrak{A}$, $\mathfrak{R}$, and $\mathfrak{h}$ are now in their initial states. At this point the desired goal is that ω should move past the remaining scattering centers (if any) without generating any changes in the $\mathfrak{A} + \mathfrak{R} + \mathfrak{h} + \mathfrak{r}$ system state.

However, this desired halting process is impossible in general for any isolated Hamiltonian system. This can be seen as follows: Any operator V which acts on an orthonormal basis $\{\psi_x | x \in X\}$ according to $V\psi_x = \psi_{F(x)}$ for some $F: X \to X$ is unitary if and only if F is a bijection. Iterations of V correspond to iterations of F.

The halting requirement is that if $\bar{x}$ corresponds to a desired final state then $\bar{x}$ must be a fix point of F. However there must also be some $x \neq \bar{x}$ such that $F(x) = \bar{x}$ since there must be some iterative entry to the desired final state. But any F which satisfies the above is at least two-one which requires a nonunitary V.

The arguments hold also for the case at hand in which $X = A \times A_b^{\{1,n\}} \times \{1, n\}$ and the iterations of V correspond to applying V'_1, [equation (25)], $V'_2 = \exp(iKH'_2)$, and V'_3 [equation (26)], in turn over and over. At least one of the operators V'_1, V'_2, or V'_3 must provide entry to the desired final state

which must be invariant for all three operators. But then the operator which provides entry is not unitary.

The solution to this dilemma, which is followed here, is to expand the system by adding yet another system $\mathfrak{B}$ which is completely described by an orthonormal basis $\{\psi_j^{\mathfrak{B}} \mid j=0,1,\ldots,n\}$. (It is convenient to use the numbers $1, 2, \ldots, n$ to label the basis states.)

The function of the $\mathfrak{B}$ system is to serve as a ballast system which "soaks up" all further system changes required by a Hamiltonian evolution once a desired final $\mathfrak{A} + \mathfrak{R} + \mathfrak{b} + \mathfrak{r}$ state has been reached. To this end let $\psi_0^{\mathfrak{B}}$ be selected as the initial state of $\mathfrak{B}$. Also let $P_0^{\mathfrak{B}}$ be the projection operator on $\mathcal{H}_{\mathfrak{B}}$ which projects out $\psi_0^{\mathfrak{B}}$. Define $P_{\neq 0}^{\mathfrak{B}}$ by $1 - P_0^{\mathfrak{B}} = P_{\neq 0}^{\mathfrak{B}}$. Let $f: \{0, n\} \to \{0, n\}$ be a bijection such that $f'(0) \neq 0$ for $0 < l \leqslant n$ and H_f a self-adjoint operator such that

$$\exp\left(iKH_f^{\mathfrak{B}}\right)\psi_l^{\mathfrak{B}} = \psi_{f(l)}^{\mathfrak{B}} \tag{31}$$

is satisfied for each l.

In the expanded model it is desired to transfer evolution to the $\mathfrak{B}$ system when $\mathfrak{A} + \mathfrak{R} + \mathfrak{b}$ has been returned to its initial state with $\mathfrak{R}$ blank. Since $\mathfrak{A} + \mathfrak{R} + \mathfrak{b}$ first enters the initial state in the erase phase at the completion of a record cell erasure with H_3' active, the next operator to be active is H_1'. Thus H_1' should be modified to contain the interactions which transfer the evolution to $\mathfrak{B}$ and keep it there. To this end define the operators h_1, h_2, and h_3 by

$$h_1 = H_1' \otimes P_0^{\mathfrak{B}} + 1^{\mathfrak{A}} \otimes P_{\mathfrak{b}}^{\mathfrak{R}} \otimes 1^{\mathfrak{b}} \otimes P_{\neq b}^{\mathfrak{r}} \otimes H_f^{\mathfrak{B}} \tag{32}$$

$$h_2 = H_2' \otimes P_0^{\mathfrak{B}} \tag{33}$$

$$h_3 = H_3' \otimes P_0^{\mathfrak{B}} \tag{34}$$

These operators along with the Hamiltonian defined by

$$H = H_0 + \sum_{j=1}^{3n} V(x - x_j) h_{j \bmod 3} \tag{35}$$

($3l \bmod 3 = 3$) represent the final model. Note that from equations (10) and (23) the terms in h_1 are orthogonal so at most one term is active during any step.

It is clear that these interaction operators do what is desired if one starts out with $\mathfrak{B}$ in the state $\psi_0^{\mathfrak{B}}$. The operator $P_{\mathfrak{b}}^{\mathfrak{R}} \otimes P_{\neq b}^{\mathfrak{r}}$ guarantees that the

$\mathfrak{B}$ system evolution does not begin until the $\mathfrak{A}$, $\mathfrak{R}$, $\mathfrak{h}$, and $\mathfrak{r}$ systems have reached the desired final states with $\mathfrak{A}$ and $\mathfrak{h}$ in their initial states, $\mathfrak{R}$ blank, and the appropriate T fix point recorded in $\mathfrak{r}$. The evolution of $\mathfrak{B}$ starts and continues with changes occurring every third step. The changes correspond to the iteration of f on 0. Nothing further happens to the $\mathfrak{A}$, $\mathfrak{R}$, $\mathfrak{h}$, and $\mathfrak{r}$ systems since $P_0^{\mathfrak{B}}\psi_{f^l(0)}^{\mathfrak{B}} = 0$. [Recall that $f^l(0) \neq 0$.]

5. APPLICATION TO TURING MACHINES

In order to apply the foregoing to Turing machines, one first notes that the system $\mathfrak{A}$ becomes the compound system $\mathfrak{L} + \mathfrak{T} + \mathfrak{j}$ and A becomes $L \times (S_b)_b^Z \times Z$. However, since the model is valid for the first n steps of the computation, the models will be restricted to standard Turing machines as discussed in Section 2. Thus, $\mathfrak{T}$ will be restricted to have $2n+1$ cells from position $-n$ to n and $\mathfrak{L}$ will have a finite set L^n of internal state labels. As discussed before, A then becomes the finite set $ID_n = L^n \times (S_b)^{\{-n,n\}} \times \{-n, n\}$.

Complete orthonormal bases for $\mathfrak{L}$, $\mathfrak{T}$, and $\mathfrak{j}$ consist of the respective sets of states $\{\psi_l^{\mathfrak{L}} | l \in L^n\}$, $\{\psi_\phi^{\mathfrak{T}} | \varphi \in S_b^{\{-n,n\}}\}$, and $\{\psi_i^{\mathfrak{j}} | i \in \{-n, n\}\}$. With these definitions, appropriate Turing machine states have the form $\Psi_{l\varphi i} = \psi_l^{\mathfrak{L}} \otimes \psi_\varphi^{\mathfrak{T}} \otimes \psi_i^{\mathfrak{j}}$ where the triple (l, φ, i) is an instantaneous description of a machine.

As noted in Section 2, to each Turing machine Q there corresponds a unique map $T_Q: ID \to ID$ defined by equation (1). Here the restriction T_{Qn} of T_Q to ID_n is considered. The T_{Qn} so defined can be taken directly into the quantum mechanical models already given in the previous section. However, for Turing machines one need not record the complete machine description (l, φ, i) into the record cells of $\mathfrak{R}$ —it is sufficient to record $(l, \varphi(i), i)$, (the state of $\mathfrak{L}$, the symbol in the $\mathfrak{T}$ cell scanned by $\mathfrak{j}$, and the position of $\mathfrak{j}$).

To this end, one replaces the sum over a in the definition of H_1 [equation (8)] by a sum over l, s, and k' to get

$$H_1 = \sum_{l \in L^n} \sum_{y \in S_b} \sum_{k'=-n}^{n} \sum_{k=1}^{n} P_l^{\mathfrak{L}} \otimes P_{yk'}^{\mathfrak{T}} \otimes P_{k'}^{\mathfrak{j}} \otimes H_{(lyk'),k}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \tag{36}$$

Here $P_l^{\mathfrak{L}}$, $P_{yk'}^{\mathfrak{T}}$, and $P_{k'}^{\mathfrak{j}}$ are projection operators for finding $\mathfrak{L}$ in state l, y in the kth cell of $\mathfrak{T}$, and the head $\mathfrak{j}$ at position k', respectively. $H_{(lyk'),k}^{\mathfrak{R}}$ is the same as $H_{ak}^{\mathfrak{R}}$ [equation (8)] with (lyk') replacing a. It is assumed that one records a triple (lyk') into each cell of $\mathfrak{R}$. Corresponding to equation (5), one has

$$V_1 = e^{iKH_1} = \sum_{l \in L^n} \sum_{y \in S_b} \sum_{k'=-n}^{n} \sum_{k=1}^{n} P_l^{\mathfrak{L}} \otimes P_{yk'}^{\mathfrak{T}} \otimes P_{k'}^{\mathfrak{j}} \otimes U_{(lyk'),k}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \tag{37}$$

where $U^{\mathfrak{R}}_{(lyk'),k} = \exp(iKH^{\mathfrak{R}}_{(lyk'),k})$. This result follows from the pairwise orthogonality of the terms in the sums of equation (36).

For the machine computation step, the operator H^Q_2 is defined similar to equation (9) by

$$H^Q_2 = \sum_{l\in L''} \sum_{s\in S} \sum_{k'=-n}^{n} \sum_{k=1}^{n} H^Q_{lsk'} \otimes P^{\mathfrak{R}}_{(lsk'),k} \otimes P^{\mathfrak{h}}_k \tag{38}$$

$P^{\mathfrak{R}}_{(lsk'),k}$ is the projection operator for finding (lsk') in cell k of $\mathfrak{R}$. Note the s sum is over s instead of S_b.

The operator $H^Q_{lsk'}$ is given by

$$H^Q_{lsk'} = H^{\mathfrak{L}Q}_{ls} + H^{\mathfrak{T}Q}_{lsk'} + H^{\mathfrak{i}Q}_{lsk'} \tag{39}$$

Let the unique quintuple in Q which begins with ls be given by $l(s, s'\sigma)m$. Then the operators $H^{\mathfrak{L}Q}_{ls}$, $H^{\mathfrak{T}Q}_{lsk'}$, and $H^{\mathfrak{i}Q}_{lsk'}$ are defined to satisfy the following: $\exp(iKH^{\mathfrak{L}Q}_{ls})$ exchanges $\psi^{\mathfrak{L}}_l$ and $\psi^{\mathfrak{L}}_m$ and leaves all other states in the basis $\{\psi^{\mathfrak{L}}_l \mid l\in L''\}$ alone. $\exp(iKH^{\mathfrak{T}Q}_{lsk'})$ exchanges $\psi^{\mathfrak{T}_k}_s$ and $\psi^{\mathfrak{T}_k}_{s'}$ only and leaves all other states in the basis $\{\psi^{\mathfrak{T}_k}_{s_1} \mid s_1\in S_b\}$ for the kth cell of $\mathfrak{T}$ alone. It has no effect on component states of other cells in $\mathfrak{T}$. Finally, $\exp(iKH^{\mathfrak{i}Q}_{lsk'})$ exchanges $\psi^{\mathfrak{i}}_{k'}$ and $\psi^{\mathfrak{i}}_{l'k'+\sigma}$ (provided that $k'\neq -n$ or $\sigma\neq -1$ and $k'\neq n$ or $\sigma\neq 1$) and leaves other states in the basis alone. For $k'=-n$ and $\sigma=-1$ or $k'=n$ and $\sigma=+1$, $\psi^{\mathfrak{i}}_{-n}$ and $\psi^{\mathfrak{i}}_n$ are exchanged and other basis states are left alone. As is the case with the other interaction operators, $H^{\mathfrak{L}Q}_{ls}$, $H^{\mathfrak{T}Q}_{lsk'}$, and $H^{\mathfrak{i}Q}_{lsk'}$ are taken to have the generic form given in equations (11)–(13) with $\Psi_1 = \psi^{\mathfrak{L}}_l$, $\psi^{\mathfrak{T}_k}_s$, and $\psi^{\mathfrak{i}}_{k'}$, respectively, and $\Psi_2 = \psi^{\mathfrak{L}}_m$, $\psi^{\mathfrak{T}_k}_{s'}$, $\psi^{\mathfrak{i}}_{k'+\sigma}$, respectively. Note that just as is the case for the recording operator, $H^{\mathfrak{T}Q}_{lsk'}$ acts on only one system (the k'th component system) which is part of a larger system ($\mathfrak{T}$).

One sees from equation (39) that the component interactions of $H^Q_{lsk'}$ are uncorrelated. This has the result that

$$\exp(iKH^Q_{lsk'}) = \exp(iKH^{\mathfrak{L}Q}_{ls}) \otimes \exp(iKH^{\mathfrak{T}Q}_{lsk'}) \otimes \exp(iKH^{\mathfrak{i}Q}_{lsk'})$$

when one uses the pairwise orthogonality of the terms in equation (38) to write

$$\exp(iKH^Q_2) = \sum_{l\in L''} \sum_{s\in S} \sum_{k'=-n}^{n} \sum_{k=1}^{n} \exp(iKH^Q_{lsk'}) \otimes P^{\mathfrak{R}}_{(lsk')k} \otimes P^{\mathfrak{h}}_k + 1 - P_2 \tag{40}$$

where

$$P_2 = \sum_{l \in L^n} \sum_{s \in S} \sum_{k'=-n}^{n} \sum_{k=1}^{n} 1^{\mathfrak{L}+\mathfrak{T}+\mathfrak{j}} \otimes P^{\mathfrak{R}}_{(lsk')k} \otimes P^{\mathfrak{h}}_{k}$$

Finally, one notes that the shift operator H_3 given by equation (10) remains unchanged except for the replacement of $1^{\mathfrak{R}}$ by $1^{\mathfrak{L}+\mathfrak{T}+\mathfrak{j}}$.

It will be recalled that the quantum mechanical models constructed here aply to standard Turing machines (Section 2). As a result, the initial and final system states are required to have the respective forms, $\psi^{\mathfrak{L}}_{l_i} \otimes \psi^{\mathfrak{T}}_{\phi_i} \otimes \psi^{\mathfrak{j}}_{0} \otimes \psi^{\mathfrak{R}}_{\mathfrak{b}} \otimes \psi^{\mathfrak{h}}_{1}$ and $\psi^{\mathfrak{L}}_{l_f} \otimes \psi^{\mathfrak{T}}_{\phi_f} \otimes \psi^{\mathfrak{j}}_{0} \otimes \psi^{\mathfrak{R}}_{\alpha} \otimes \psi^{\mathfrak{h}}_{k}$. Here $\phi_i(j)$ and $\phi_f(j) = b$ for $j \leqslant 0$ and contain no blanks between nonblank symbols, α is a computation history sequence where $\alpha(j) \in L^n \otimes S_b \otimes \{-n, n\}$ for $j = 1, \ldots, k-1$ and $\alpha(j) = b$ for $k \leqslant j < n$. The above final state is reached at the end of some step j where $0 = j \bmod 3$.

For application of the general model constructions to Turing machines, the copy phase will be changed from a single step to satisfy the requirement (Bennett, 1973) that the copying also be carried out by a copying Turing machine. As a result, the model system will be expanded by adding a copy tape system $\mathcal{C}$, a copy head δ, and a spin-1/2 system ν. $\mathcal{C}$ is a lattice of tape cell systems which duplicates $\mathfrak{T}$ on $\{0, n\}$ (the extension to $\{-n, n\}$ is not necessary). $\mathcal{C}$ is completely described by the basis set $\{\psi^{\mathcal{C}}_{\phi} | \phi \in (S_b)^{\{0,n\}}\}$. δ is similar to $\mathfrak{h}$ and is described by similar states $\{\psi^{\delta}_{j} | j = 0, \ldots, n\}$. The system ν is added to distinguish between the forward and copy phase and the erasure phase. Finally, two states c and d in L^n which are not used in the forward phase are singled out as states of $\mathfrak{L}$ for the copy phase.

The expanded system starts with $\mathfrak{L} + \mathfrak{T} + \mathfrak{j} + \mathfrak{R} + \mathfrak{h}$ in an appropriate standard initial state, $\mathcal{C}$ in a state $\psi^{\mathcal{C}}_{\beta}$ with all cells blank, δ at position 0 (in state ψ^{δ}_{0}), and ν with spin up (+). If and when the computation is finished, the system enters the copying phase by changing $\psi^{\mathfrak{L}}_{l_f}$ to $\psi^{\mathfrak{L}}_{c}$. Copying is done by moving the head systems $\mathfrak{j}$ and δ in tandem down the respective tapes. When the copying is completed (signified by $\mathfrak{j}$ reaching a blank cell), the spin of ν is changed to down (−) and the system enters the erasure phase. However, in the first part of the erasure phase, the $\mathfrak{L} + \mathfrak{T} + \mathfrak{j} + \mathcal{C} + \delta$ interaction must be modified so $\mathfrak{j}$ and δ are returned in tandem to their initial positions 0 without erasing $\mathcal{C}$.

The construction given here also differs from that of Bennett (1973) in that the recording and $\mathfrak{h}$ shift steps continue to operate during the copy phase. This is not necessary since the copy machine is reversible (Bennett, 1973), however, the interaction operators have a simpler form if these steps continue to operate as they do not have to be turned on and off.

The changes in the interaction operators which incorporate the above expansion are as follows: instead of H_1' and H_3' defined as in equations (23) and (24), one has

$$H_1' = H_1 \otimes 1^{\mathcal{C}+\delta} \otimes P_+^{\nu} + H_3 \otimes 1^{\mathcal{C}+\delta} \otimes P_-^{\nu} \tag{41}$$

and

$$H_3' = H_3 \otimes 1^{\mathcal{C}+\delta} \otimes P_+^{\nu} + H_1 \otimes 1^{\mathcal{C}+\delta} \otimes P_-^{\nu} \tag{42}$$

where H_1 and H_3 are given by equations (36) and (10), respectively, and P_+^{ν} and P_-^{ν} are the respective ν system spin-up and spin-down projection operators.

Most of the changes occur in the operator H_2'. One now has

$$H_2' = H_2^F + H_2^c + H_2^v \tag{43}$$

where

$$H_2^F = H_2^Q \otimes 1^{\mathcal{C}+\delta+\nu} + \sum_{s \in S} \sum_{k'=-n}^{n} \sum_{k=1}^{n} H_{cl_f}^{\mathfrak{L}} \otimes 1^{\mathfrak{T}+1} \otimes P_{(l_f s k'),k}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \otimes 1^{\mathcal{C}+\delta+\nu} \tag{44}$$

and

$$H_2^v = \sum_{k'=-n}^{n} \sum_{k=1}^{n} 1^{\mathfrak{L}+\mathfrak{T}+1} \otimes \left[P_{(cbk'),k}^{\mathfrak{R}} + P_{(l_f bk'),k}^{\mathfrak{R}} \right] \otimes P_k^{\mathfrak{j}} \otimes 1^{\mathcal{C}+\delta} \otimes H_{+-}^{\nu} \tag{45}$$

Here, H_2^Q is given by equation (38), H_{cl_f} is the interaction operator for the exchange of $\psi_{l_f}^{\mathfrak{L}}$ and $\psi_c^{\mathfrak{L}}$, and $\exp(iKH_{+-}^{\nu})$ exchanges ψ_+^{ν} and ψ_-^{ν}. The change to the copy phase state $\psi_c^{\mathfrak{L}}$ occurs if and only if the final $\mathfrak{L}$ state label l_f appears in a record cell scanned by $\mathfrak{h}$. The ν system spin projection change occurs if and only if cb or $l_f b$ appears in a record cell scanned by $\mathfrak{h}$. The appearance of cb indicates that the copying is complete and $l_f b$ is included to take care of the possibility that the forward phase computation ends with all cells of $\mathfrak{T}$ blank. In this case, no copying is necessary.

Furthermore, all terms in the k', and k sums of equation (45) are pairwise orthogonal as are all terms in the ℓ, s, k', and k sums of equations (38) and (44). In particular, for $l = l_f$ terms in equation (38), $H_{l_f s k'}^Q = 0$ for all s and k'. Thus, at most one of the terms is active at any model step.

The copy interaction operator H_2^c is more complex as one must give here explicit expressions for the interaction operators for each type of step in the copy and initial erase phase. One has

$$H_2^c = H_2^1 + H_2^2 + H_2^3 + H_2^4 \tag{46}$$

where

$$H_2^1 = \sum_{s \in S} \sum_{k'=-n}^{n} \sum_{k=1}^{n} H_{cd}^{\mathfrak{L}} \otimes 1^{\mathfrak{T}+\mathfrak{j}} \otimes \left[P_{(csk'),k}^{\mathfrak{R}} + P_{(dsk'),k}^{\mathfrak{R}} \right] \otimes P_k^{\mathfrak{h}} \otimes 1^{\mathfrak{C}+\delta+\nu} \tag{47}$$

$$H_2^2 = \sum_{s \in S} \sum_{k'=-n}^{n} \sum_{k=1}^{n} 1^{\mathfrak{L}+\mathfrak{T}+\mathfrak{j}} \otimes P_{(csk'),k}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \otimes H_{sk'}^{\mathfrak{C}} \otimes P_{k'}^{\delta} \otimes P_+^{\nu} \tag{48}$$

$$H_2^3 = \sum_{s \in S} \sum_{k'=-n}^{n} \sum_{k=1}^{n} 1^{\mathfrak{L}+\mathfrak{T}} \otimes \left[H_{k'+1}^{\mathfrak{j}} \otimes P_{(dsk'),k}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \otimes 1^{\mathfrak{C}+\delta} + 1^{\mathfrak{j}} \otimes P_{(dsk'),k}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \otimes H_{k'+1}^{\delta} \right] \otimes P_+^{\nu} \tag{49}$$

$$H_2^4 = \sum_{s \in S} \sum_{k'=-n}^{n} \sum_{k=1}^{n} 1^{\mathfrak{L}+\mathfrak{T}} \otimes \left[H_{k'-1}^{\mathfrak{j}} \otimes P_{(dsk'),k}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \otimes 1^{\mathfrak{C}+\delta} + 1^{\mathfrak{j}} \otimes P_{(dsk'),k}^{\mathfrak{R}} \otimes P_k^{\mathfrak{h}} \otimes 1^{\mathfrak{C}} \otimes H_{k'-1}^{\delta} \right] \otimes P_-^{\nu} \tag{50}$$

Here H_2^1 is the interaction operator for exchanging the copy phase states $\psi_c^{\mathfrak{L}}$ and $\psi_d^{\mathfrak{L}}$ of $\mathfrak{L}$. The projection operator sum in the brackets is over the two states c and d being recorded (along with the position of $\mathfrak{j}$ and the contents of the $\mathfrak{T}$ cell scanned by $\mathfrak{j}$) in the $\mathfrak{R}$ cell scanned by $\mathfrak{h}$.

H_2^2 is the operator which copies the expression in $\mathfrak{T}$ into $\mathfrak{C}$. In particular, it records the symbol s into the k'th $\mathfrak{C}$ cell provided that δ is at position k' and (csk') is in the $\mathfrak{R}$ cell scanned by $\mathfrak{h}$. By the model construction, one can base the copying on what is in the record rather than on the actual $\mathfrak{T}$ state because at each H_2^c copy step, the scanned record cell reflects the position of $\mathfrak{j}$ and the contents of the $\mathfrak{T}$ cell scanned by $\mathfrak{j}$. $H_{sk'}^{\mathfrak{C}}$ is the interaction operator such that $\exp(iKH_{sk'})$ exchanges s and b in the k'th cell only of $\mathfrak{C}$ and leaves the k'th cell alone otherwise. All other cells of $\mathfrak{C}$ are also left alone.

H_2^3 is the operator that shifts both the $\mathfrak{j}$ and δ heads one step to the right and H_2^4 shifts the two heads one step to the left.

In understanding how H_2^1 operates, one first notes that H_2^c and H_2^F are orthogonal because the $\mathfrak{L}$ state labels in the record cell projection operators in H_2^F are different from c or d (which are the only labels appearing in H_2^c). Also H_2^c and $H_2^{\prime\prime}$ are orthogonal because the only $\mathfrak{T}$ cell symbol appearing in the record cell projection operator is the blank while the corresponding s sums in H_2^c are over S and not S_b.

Within H_2^c, one has, as before, that terms with different values of s, k', or k are orthogonal. However for each (s, k', k), the operator $H_{cd}^{\mathfrak{L}}$ in H_2^1 acts, along with the appropriate factors in H_2^2, H_2^3, and H_2^4, to give the interaction operators $H_{cd}^{\mathfrak{L}} + H_{sk'}^{\mathfrak{C}}$, $H_{cd}^{\mathfrak{L}} + H_{k'+1}^{\mathfrak{j}} + H_{k'+1}^{\delta}$, and $H_{cd}^{\mathfrak{L}} + H_{k'-1}^{\mathfrak{h}} + H_{k'-1}^{\delta}$, respectively. These remain in the exponent of $\exp(iKH_2^c)$ after one brings down the projection operators [e.g., as in equation (40)]. At most, one of these interaction operators is active in any step since H_2^2, H_2^3, and H_2^4 are pairwise orthogonal.

Further details on H_2' and in particular, the proof that H_1', H_2', and H_3' act as they are claimed to act, particularly during the copy phase and the first part of the erase phase, are left to the reader as they are straightforward.

The final model is obtained by addition of a ballast system $\mathfrak{B}$ just as in the general case. When and if the model arrives back at a state which is initial for $\mathfrak{L}$, $\mathfrak{T}$, $\mathfrak{j}$, $\mathfrak{R}$, $\mathfrak{h}$, and δ, that is,

$$\psi_{l_i}^{\mathfrak{L}} \otimes \psi_{\phi}^{\mathfrak{T}} \otimes \psi_0^{\mathfrak{j}} \otimes \psi_{\beta}^{\mathfrak{R}} \otimes \psi_1^{\mathfrak{h}} \otimes \psi_{\alpha}^{\mathfrak{C}} \otimes \psi_0^{\delta} \otimes \psi_{(-)}^{\prime\prime} \tag{51}$$

where ϕ corresponds to the initial $\mathfrak{T}$ tape expression starting at 0, β is the blank sequence, and α corresponds to the final $\mathfrak{T}$ tape expression, then all further evolution is transferred to the $\mathfrak{B}$ system. The final Turing machine model Hamiltonians are given by equation (32)–(35) with H_1', H_2', and H_3' given by equations (41)–(43). In equation (32) $P_{\neq b}^{\mathfrak{C}}$, the projection operator for at least one cell of $\mathfrak{C}$ being nonblank, replaces $P_{\neq b}^{\mathfrak{r}}$. Since for these models the ballast system evolution is exactly the same as discussed in the last section, it will not be repeated here.

6. DISCUSSION

There are several points about the models which should be noted. The quantum mechanical Hamiltonian models are constructed so that they evolve as pure states in isolation from their surroundings. By well-known principles, the energy of the complete system is independent of time. Since the overall system state is pure, the overall system entropy is zero and remains zero throughout the $3n$ model steps.

However, in common with other systems, the model system state described here does degrade. To see this, consider increasing the number $3n$ of scattering centers in $\mathcal{C}$ while keeping the other system parameters fixed. For each time t and ω position x, the overall system wave function given by equations (16) and (19) is a good approximation to the exact wave function provided that $n(x)$ satisfies the limitations given by equations (20), (21), and (22) with $n(x)$ replacing $3n$ in these equations. In particular, this means that the components of the exact wave function which are discarded in making the eikonal approximation and in deriving equation (17) from the resulting Lippman–Schwinger equation are small. Similarly, the components discarded by replacing the more exact equation (17) by the less exact equation (19) are small.

Even though these undesirable components are small, they slowly increase with increasing time and step number. For times and step numbers which do not satisfy the limitations given by equations (20)–(22), these components may not be small compared to the desired state.

It is clear then that as the exact state vector evolves, the amplitude of the undesired components increases and that of the desired component decreases. Furthermore, the undesired components do not describe the desired ordered and completed development of each step in the modeled process. It is in this sense that one can speak of the state vector degrading.

Of course, the degradation is slow. As noted in the example, neglecting limitations on the interactions between the $\mathfrak{A}$, $\mathfrak{R}$, $\mathfrak{h}$, $\mathfrak{r}$, and $\mathfrak{B}$ systems, the model describes 10^6 model steps taking place in 10^{-7} sec before degradation becomes appreciable. Furthermore, the degradation can be slowed down by changing the system parameters so that the model is good for more than 10^6 steps. The price one pays for this is that the time taken per step must increase with the result that the model step evolution slows down.

It is also true that, contrary to what was implied in earlier work (Benioff, 1980, 1981), energy is dissipated within the overall system. In particular, as amplitudes of undesired components, which represent evolution in undesired directions, increase, so does the energy associated with these components. Since the total energy is fixed, the energy dissipates from the desired component into the undesired ones. (There is no dissipation to external sinks as the overall system is isolated.)

As an example, one can consider the effect of approximating equation (17) by equation (19). Use of the more exact equation (17) shows undesirable components growing in due to deviations from K of the exponent coefficient D/k. In particular, one can show that for equation (17), the energy dissipation increases linearly with the step number and, for the Hamiltonians of the form used here, equations (11)–(13), the energy dissipation per step is about equal to $[\hbar^2(\Delta k)^2/2m]\,(\pi^2/4)$ where $(\Delta k)^2$ is the

dispersion of $\phi(k-k_0)$ in equation (16). No such dissipation occurs if the more approximate equation (19) is used.

The description of the various systems such as $\mathfrak{A}$, or $\mathfrak{L}$, $\mathfrak{T}$, j and $\mathfrak{R}$, $\mathfrak{h}$, and $\mathfrak{C}$ or r was given in the abstract. If desired, the tape heads can be represented as spinless particles on a lattice and the other systems can be represented as lattices of spin-1/2 systems. These representations require one–one maps from the corresponding states to the set of 0–1 sequences, where 0 and 1 correspond to spin up and spin down, respectively.

For example, $\mathfrak{L}$ can be a finite one-dimensional lattice of length $\ln_2(N_n)$ and $\mathfrak{T}$ could be a two-dimensional lattice of length $2n-1$ in one dimension (each unit represents a tape cell) and of length $\ln_2(m)$ in the other dimension where $m=$number of symbols in S_b. $\mathfrak{R}$ and $\mathfrak{C}$ can also be represented as finite two-dimensional lattices where the degrees of freedom in one dimension refer to the cell positions, and in the other dimension, they refer to the distinct symbols to be recorded. Since the number $(2n+1)mN_n$ of triples in $L^n \otimes S_b^{\otimes}\{-n, n\}$ is finite, each record cell could be a lattice of length $\ln_2[(2n+1)mN_r]$.

Another interesting point concerns the existence of universal Turing machines, i.e., those which mimic any Turing machine. The fact that such machines exist means that there exists an interaction operator H_2, defined by equation (38), such that one fixed operator H_2' defined by equations (43)–(50) is sufficient to model all Turing machines. The differences between different machines, instead of being reflected as differences between model Hamiltonians, is thus transferred to differences between initial states, i.e., what is on the tape $\mathfrak{T}$.

In the model constructed here, if the process reaches a fix point in a small enough number of steps then at the end of $3n$ steps the object system $\mathfrak{A}+\mathfrak{R}+\mathfrak{h}$ (or $\mathfrak{L}+\mathfrak{T}+\mathfrak{j}+\mathfrak{R}+\mathfrak{h}$ in the case of Turing machines) is back in its initial state. However, the ballast system $\mathfrak{B}$ is not. In order to restore the ballast system to its initial state one must dissipate energy in the amount of $kT\ln_2 n$. This follows from the fact that for the model considered here $\mathfrak{B}$ must have at least n distinct states.

This dissipation is an example of the fact noted by Landauer and others (Keyes and Landauer, 1970; Landauer and Woo, 1971; Landauer, 1976) that whenever one destroys information, one dissipates kT energy per bit destroyed. However, this energy expenditure does not take place during the process. It occurs at the end of the process if and only if one restores $\mathfrak{B}$ to its initial state. Note that it is proportional to $\ln_2 n$ and not to n.

Finally, it is to be stressed that what has been shown here is the mathematical existence of quantum mechanical Hamiltonian models of systems (including Turing machines) that erase their own histories. Whether or not such models are actually physically constructible is an open question.

In regard to this question, one notes that in common with many other quantum mechanical models, the models constructed here describe isolated systems. This represents an idealized state of affairs which is only realized approximately. Also, the models constructed here are very sensitive to any external influences. The reason is that all components of the object system are noninteracting during times the step interactions are not active. The object system energy at the end of each step is independent of the step number and equals zero.

This open question is closely related to a more basic one. Which Hamiltonian evolutions are constructible in the sense that there is a well-defined laboratory procedure for arranging systems and external fields so that the systems time evolution is given by the Hamiltonian under consideration? Simple cardinality arguments suggest that most Hamiltonians are not physically constructible just as most quantum mechanical observables are not measureable. In this case, are the Turing machine Hamiltonians physically constructible or not? It is likely that answers to questions such as these will have to await more work.

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